Received: *April 25, 2000*, Accepted: *June 26, 2000*

On the stringy nature of winding modes in non-commutative thermal field theories

Giovanni Arcioni*

Spinoza Instituut, Universiteit Utrecht, Leuvenlaan 4
3584 CE Utrecht, The Netherlands
E-mail: arcioni@to.infn.it, g.arcioni@phys.uu.nl

Jose L.F. Barbón and Joaquim Gomis†

Theory Division, CERN, CH-1211 Geneva 23, Switzerland
E-mail: barbon@mail.cern.ch, gomis@mail.cern.ch, gomis@ecm.ub.es

Miguel A. Vázquez-Mozo

Instituut voor Theoretische Fysica, Universiteit van Amsterdam
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
M.Vazquez-Mozo@phys.uu.nl

Abstract: We show that thermal non-commutative field theories admit a version of "channel duality" reminiscent of open/closed string duality, where non-planar thermal loops can be replaced by an infinite tower of tree-level exchanges of effective fields. These effective fields resemble closed strings in three aspects: their mass spectrum is that of closed-string winding modes, their interaction vertices contain extra moduli, and they can be regarded as propagating in a higher-dimensional "bulk" space-time. In non-commutative models that can be embedded in a D-brane, we show the precise relation between the effective "winding fields" and closed strings propagating off the D-brane. The winding fields represent the coherent coupling of the infinite tower of closed-string oscillator states. We derive a sum rule that expresses this effective coupling in terms of the elementary couplings of closed strings to the D-brane. We furthermore clarify the relation between the effective propagating dimension of the winding fields and the true codimension of the D-brane.

Keywords: D-branes, Non-Commutative Geometry.

*Dipartimento di Fisica Teorica, Università di Torino, via P. Giuria 1, I-10125 Torino, Italy and INFN Sezione di Torino

†Departament ECM, Facultat de Fisica, Universitat de Barcelona and Institut de Fisica d'Altes Energies, Diagonal 647, E-08028 Barcelona, Spain

JHEP06(2000)038

Contents

1. Introduction **1**

2. Winding modes in non-commutative quantum field theory: an elementary example **3**

3. Integrating out a general loop **6**
3.1 Generic one-loop diagram in non-commutative ϕ^n theory 6
3.2 Some special cases at two loops 9

4. Windings and closed strings **10**
4.1 Heuristic considerations 10
4.2 Open-string channel 13
4.3 The closed-string channel sum rule 15
4.4 An illustrative example 18

5. Concluding remarks **20**

1. Introduction

It has been realized that non-commutative field theories (NCFTs) emerge as effective field theories of string/M-theory compactifications in the presence of constant antisymmetric tensor fields [1, 2, 3]. This result has triggered a renewed interest in the study of both perturbative [4, 5, 6] and non-perturbative [7] aspects of non-commutative field theories. The stringy connection of NCFTs opens up the interesting possibility of trying to understand some of their physical features by embedding them in string theory. In particular a number of NCFTs have been obtained as a low-energy limit of open-string theories in B-field backgrounds [8]–[13]. As a matter of example, one can try to understand the nonlocality inherent in these quantum field theories in terms of string theory after an appropriate low-energy limit is taken [3].

Among the most intriguing features of NCFT is a peculiar mixing between infrared and ultraviolet scales [5, 6]. On physical grounds it can be understood as the result of the uncertainty principle between two non-commuting spatial dimensions, since probing ultraviolet physics in one direction leads to infrared effects in the other. At a more technical level, this mixing reflects itself in the appearance of extra poles at zero momentum in some amplitudes, in the limit where the ultraviolet cut-off is sent to infinity. The authors of refs. [5, 6] interpreted these poles as resulting from the interchange of a new field ψ with kinetic kernel $-\partial \circ \partial \equiv \partial_\mu (\theta^2)^{\mu\nu} \partial_\nu$. It would be very interesting to see if there is a stringy interpretation for these particles.

One of the obvious ways of spotting a stringy behaviour in NCFT would be to look at situations where the presence of extended objects is made manifest, as for example studying these theories in spaces with non-trivial topology or at finite temperature [14, 15, 16]. In refs. [17, 18] it was pointed out that the two-loop thermal partition function of some NCFTs can be cast in a way that indicates the presence of states whose energy scales with the inverse temperature as $|\ell\beta|$, with ℓ some integer number. This would suggest that NCFTs contain certain extended degrees of freedom that are able to wrap around the euclidean time.

In this paper we will try to understand whether some kind of winding modes can be identified in non-commutative thermal perturbation theory, extending on the work of [17]. Actually, we shall see that the winding modes formally identified in thermodynamical quantities can be associated to effective fields with special propagators, in much the same fashion as the ψ-fields of refs. [5, 6]. In fact, the UV/IR interpretation of these propagators is the same once we realize that the temperature acts as an ultraviolet cutoff in the field theory.

Therefore, this raises the question of whether the "winding fields" could be interpreted as "off-brane" closed-string modes that survive the Seiberg-Witten (SW) decoupling limit. We find that this expectation is not fulfilled, at least in a literal sense. In particular, any closed-string picture amounts to the exchange of the infinite tower of closed-string excitations in the bulk, and therefore it is not a very transparent way of describing the dynamics. Instead, each winding field describes a sort of coherent exchange of an infinite number of closed-string modes. One of our results is the derivation of a sum rule for the effective coupling of the winding fields, in terms of the elementary couplings of closed strings to a D-brane. In fact, the interactions of these winding fields are *not* specified solely in terms of standard interaction vertices, except in very special kinematical situations. Generically, the vertices contain additional modular parameters that must be integrated over.

The paper is organized as follows. In sections 2 and 3 we extend the analysis of refs. [17, 18] to more general diagrams in NCFT at finite temperature and try to cast the loop amplitudes in a "dual channel" picture, in terms of tree-level exchanges. In

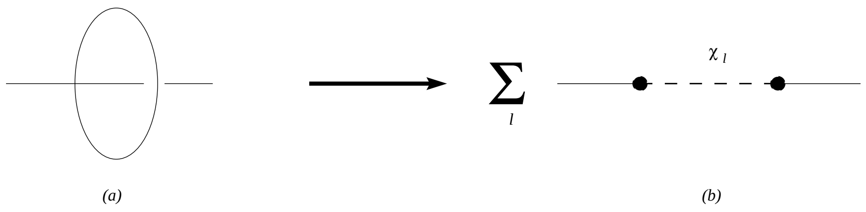

Figure 1: "Channel duality" in the non-planar self-energy diagram in ϕ^4 theory.

section 4 we will obtain these amplitudes by studying the low energy SW limit from string theory in order to identify the low-energy winding modes with undecoupled winding strings. Finally in section 5 we will summarize our conclusions.

2. Winding modes in non-commutative quantum field theory: an elementary example

The simplest situation where one can formally identify "winding modes" is that of the two-loop contribution to the free energy in a ϕ^4 theory. The planar diagram is independent of the deformation parameter $\theta^{\mu\nu}$, but a non-trivial phase $\theta(p,q) = p_\mu \theta^{\mu\nu} q_\nu$ enters the loop integral in the non-planar case[1]

$$\mathcal{F}_{\rm NP} = -g^2 \sum\!\!\!\!\!\!\!\int_p \sum\!\!\!\!\!\!\!\int_q \frac{e^{i\theta(\mathbf{p},\mathbf{q})}}{(p^2+M^2)(q^2+M^2)}\,, \tag{2.1}$$

where we have used the notation

$$p^2 = \mathbf{p}^2 + \frac{4\pi^2 n^2}{\beta^2}\,, \qquad \sum\!\!\!\!\!\!\!\int_p \equiv \frac{1}{\beta}\sum_{n\in\mathbb{Z}} \int \frac{d\mathbf{p}}{(2\pi)^{d-1}}\,.$$

The ultraviolet divergences of this integral can be appropriately eliminated by renormalization of the $T=0$ limit, as usual in thermal field theory [19]. In fact, the ultraviolet structure of this diagram is milder than that of the planar counterpart, because the divergence contributed by one of the loops is effectively cut off by the non-commutative phase provided $(\theta p)^2 \equiv (\theta^{\mu\nu} p_\nu)^2$ is non vanishing. This is an example of the UV/IR mixing of [5], namely this divergence will reappear, disguised as an infrared effect, as $\theta p \to 0$.

In ref. [17] it was pointed out that one could "integrate out" one of the loops and replace it by a statistical sum over objects living at the formally T-dual temperature $1/(\theta T)$, thus representing analogues of winding modes. The essential phenomenon can be understood by simply looking at the one-loop self-energy tadpole diagram (figure $1a$):

$$\Pi(\beta,\mathbf{p})_{\rm NP} = -g^2 \sum\!\!\!\!\!\!\!\int_q \frac{e^{i\theta(\mathbf{p},\mathbf{q})}}{q^2+M^2}\,. \tag{2.2}$$

[1] We will assume throughout that $\theta^{0i}=0$.

JHEP06(2000)038

Introducing a Schwinger-parameter representation of the propagator,

$$\Pi(\beta, \mathbf{p})_{\rm NP} = -g^2 \int_0^\infty dt \sum\!\!\!\!\!\!\!\int_q e^{-t\left(q^2+M^2-i\theta(\mathbf{p},\mathbf{q})/t\right)}\,, \tag{2.3}$$

we can perform the gaussian integral over $\mathbf{q}$. After a further Poisson resummation in the thermal frequency running in the loop we obtain

$$\Pi(\beta, \mathbf{p})_{\rm NP} = -\frac{g^2}{4\pi^{d/2}} \int_0^\infty ds\, s^{\frac{d-4}{2}} \sum_{\ell\in\mathbb{Z}} e^{-s[\beta^2\ell^2+(\theta\mathbf{p})^2]-M^2/4s}\,, \tag{2.4}$$

where we have changed variables to the "dual" Schwinger parameter $s = 1/4t$. This form is very convenient to perform the subtraction of the $T = 0$ self-energy, since we simply have to restrict the integer sum to $\ell \neq 0$.

For $d < 4$, the explicit power of $s^{\frac{d-4}{2}}$ in the proper time integral can be "integrated in" into the exponent by introducing $4 - d$ extra gaussian variables $\mathbf{z}_\perp$, and we can write the full non-planar loop in the following suggestive form:

$$\Pi(\beta, \mathbf{p})_{\rm NP} = -\sum_{\ell\in\mathbb{Z}} \int d\mathbf{z}_\perp \frac{|g_{\phi\chi}(\ell, \mathbf{p}, \mathbf{z}_\perp)|^2}{\beta^2\ell^2 + (\theta\mathbf{p})^2 + \mathbf{z}_\perp^2}\,. \tag{2.5}$$

In other words, we have written the original loop diagram in a "dual channel" in terms of an infinite number of tree-level exchanges of particles χ_ℓ with momenta $\theta\mathbf{p}$, mass proportional to $|\beta\ell|$ and extra momentum variables in $d_\chi^\perp = 4-d$ transverse dimensions (figure 1b). This complete expression renormalizes the mass of the particle running in the second loop.[2]

The mass of the χ_ℓ-fields, scaling as integer multiples of the thermal length, is characteristic of winding modes of closed strings. The effective coupling squared of these particles to the fields in external legs is given by

$$|g_{\chi\phi}(\ell, \mathbf{p}, \mathbf{z}_\perp)|^2 = \frac{g^2}{4\pi^2} \int_0^\infty ds\; e^{-s-\frac{1}{4s}M^2[\beta^2\ell^2+(\theta\mathbf{p})^2+\mathbf{z}_\perp^2]}\,. \tag{2.6}$$

Thus, if the original field was massive, the coupling to the χ_ℓ-field is suppressed at high values of momentum and winding number ℓ, i.e. only fields with winding numbers $|\ell| < (\beta M)^{-1}$ contribute significantly to the tree-level exchange. The most interesting case is that of a massless field theory. In this case the effective coupling is constant and weights all winding numbers democratically, with the coupling strength $g_{\phi\chi} = g/(2\pi)$. Finally, if the ϕ^4-field is tachyonic, the whole expression is meaningless, since it diverges at the $s = 0$ end. For this matter this "channel duality" in NCFT is reminiscent of open/closed-string channel duality in string theory. Since NCFTs

[2] One could proceed in the standard way and perform a resummation of ring diagrams.

can be obtained in many cases as low-energy limits of open-string theories, we find it natural that the "dual channel", obtained through a modular transformation $t = 1/4s$, exhibits the open-string tachyon as an ultraviolet divergence.

It is most interesting to compare the winding χ_ℓ-fields we have defined with the ψ-particles of [5, 6]. The structure of the propagator shows that these fields are formally similar

$$\langle \chi_\ell(-\mathbf{p}, -\mathbf{z}_\perp)\, \chi_\ell(\mathbf{p}, \mathbf{z}_\perp)\rangle = \frac{1}{\beta^2\ell^2 + (\theta\mathbf{p})^2 + \mathbf{z}_\perp^2}\,, \tag{2.7}$$

namely, they have a "static" kinetic term with the kernel $-\partial \circ \partial = (\theta\mathbf{p})^2$ for a field of non-canonical dimension. Furthermore, at least as long as $d \leq 4$, the non-standard power of the propagator can be understood in terms of a free propagation in a $4-d$ dimensional "transverse bulk". The effective mass $|\beta\ell|$ plays also the role of the inverse ultraviolet cutoff Λ^{-1} in the treatment of [5] and, in the absence of the explicit ultraviolet cutoff, the original ultraviolet divergence is back as an infrared divergence at $\theta\mathbf{p} \to 0$. In our expression, this shows up as a pole in the zero-winding sector. It is precisely this contribution that is subtracted when renormalizing the self-energy by the zero-temperature one.[3] Therefore, we confirm that $|\beta\ell|$ plays the role of a regulator.

One important difference between our tree-level exchange interactions and the ψ-fields of [5, 6] is that our ultraviolet cutoff T has a physical interpretation, and we are free from the arbitrariness of the choice of Wilsonian cutoffs. In particular we can integrate out the complete non-planar loop in terms of the infinite tower of tree exchanges of χ_ℓ particles. The manipulation is not *a priori* restricted to the extreme ultraviolet part.

One interesting aspect of the tree-exchange "dual" representation (2.5) is that it admits an interpretation for the planar diagram too. The only difference in the planar case comes from setting $\theta\mathbf{p} = 0$. Therefore, the planar thermal loop can be replaced in this case by

$$\Pi(\beta, \mathbf{p})_{\rm P} - \Pi(\infty, \mathbf{p})_{\rm P} = -\sum_{\ell \neq 0} \int d\mathbf{z}_\perp \, \frac{|g_{\chi\phi}(\ell, \mathbf{p}=0, \mathbf{z}_\perp)|^2}{\beta^2\ell^2 + \mathbf{z}_\perp^2}\,. \tag{2.8}$$

Now we must work with the fully renormalized quantity ($\ell \neq 0$ in the winding sum) and the propagator of the χ_ℓ particles is inserted formally at zero non-commutative momentum, i.e. we have a sum over zero-momentum tadpoles of the χ_ℓ-fields. This

[3] Notice that this procedure is different from the one followed in [17] where the authors worked with the two-loop free energy for ϕ^4 NCFT before subtracting the zero-temperature counterterms. In that case the ultraviolet divergence in one of the original loops partially transforms into an infrared one after Poisson resummation and integration over the loop momenta. This is just a consequence of UV/IR mixing.

JHEP06(2000)038

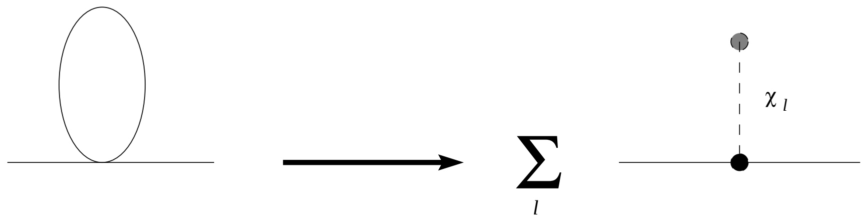

Figure 2: Dual-channel interpretation of the thermal loop in the planar contribution to the two-point function in ϕ^4 non-commutative field theory.

is also reminiscent of the closed-string interpretation, because closed strings have tree-level tadpoles on D-branes. Now we would be inclined to interpret the residue of the propagator poles as the product of the couplings $g_{\chi\phi\phi} \cdot g_{\chi\text{-vac}}$ (figure 2).

3. Integrating out a general loop

In the above example we have seen how the effect of a thermal tadpole loop in a non-commutative ϕ^4 theory admits a "dual channel" interpretation in terms of a tree-level exchange of some "winding" χ_ℓ-field with inverse propagator $\beta^2\ell^2 + (\theta\mathbf{p})^2$ which mixes with the fundamental ϕ-quantum. It would be interesting to decide to what extent this duality between thermal loops and tree-level χ_ℓ-exchanges is a general feature of NCFT at finite temperature, or just a property of a particular class of diagrams and theories.

3.1 Generic one-loop diagram in non-commutative ϕ^n theory

The first case we can consider is the generalization of the example studied in the previous section, a generic one-loop diagram with $N = N_+ + N_-$ vertices in the non-commutative version of ϕ^n field theory in d dimensions (figure 3), where we will take N_- vertices as "twisted", so the amplitude will be non planar whenever $N_\pm \neq N$. Thus, the fully amputated amplitude can be written as

$$\mathcal{A}(p_1, \ldots, p_N) = g^{2N}\, \mathcal{W}_{\rm NC} \sum\!\!\!\!\!\!\!\int_q \prod_{a=1}^{N} \frac{e^{-\frac{i}{2}\xi_a p_a \theta\, q}}{(q+Q_a)^2 + M^2}\, \delta(Q_N)\,,$$

where $Q_a = \sum_{i=1}^{a} p_i$ and $\xi_a = \mp 1$ depending on whether the insertion is twisted or not; p_a indicates the total momentum entering the loop through the a-th insertion and $\mathcal{W}_{\rm NC}$ stands for the global non-commutative phase of the diagram, depending only on the external momenta. For the sake of simplicity we will consider here "global" twists where all the $n-2$ external legs entering the loop through the same insertion have the same twist ξ_a. The generalization to include different twist in the same insertion is straightforward and does not affect the conclusions of our analysis.

Using Feynman and Schwinger parameters we can write

$$\mathcal{A}(p_1,\ldots,p_a) = g^{2N}(N-1)!\,\mathcal{W}_{\rm NC}\int_0^\infty dt\,t^{N-1}\,e^{-tM^2}\int_0^1 [dx]\,e^{-t\sum_a x_a Q_a^2}\times \\ \times\frac{1}{\beta}\sum_{n\in\mathbb{Z}} e^{-t\left(\frac{4\pi^2n^2}{\beta^2}+\frac{4\pi n}{\beta}\sum_a x_a Q_a^0\right)}\int\frac{d\mathbf{q}}{(2\pi)^{d-1}}e^{-t(\mathbf{q}^2+2\mathbf{q}\cdot\sum_a x_a\mathbf{Q}_a)}e^{i\mathbf{p}_{\rm np}\theta\mathbf{q}}\,, \tag{3.1}$$

the integration measure $[dx]$ over the Feynman parameters x_a $(a=1,\ldots,N)$ is given by

$$[dx]\equiv\delta\left(\sum_{a=1}^N x_a-1\right)\prod_{a=1}^N dx_a$$

and $\mathbf{p}_{\rm np}$ denotes the total *non-planar* spatial momentum entering the loop through the N_- "twisted" insertions, $\mathbf{p}_{\rm np}\equiv-\frac{1}{2}\sum_{a=1}^N\xi_a\mathbf{p}_a$.

By integrating the loop spatial momentum and performing a Poisson resummation the total amplitude can be recast in terms of the dual Schwinger parameter $s=1/(4t)$ in the form

$$\mathcal{A}(p_1,\ldots,p_N) = g^{2N}\frac{(N-1)!}{2^{2N}\pi^{\frac{d}{2}}}\mathcal{W}_{\rm NC}\times \\ \times\int_0^\infty ds\,s^{\frac{d-2N-2}{2}}\sum_{\ell\in\mathbb{Z}}e^{-s\left[\beta^2\ell^2+(\theta\mathbf{p}_{\rm np})^2\right]}F_\ell(s;\beta,p_1,\ldots,p_N)\,, \tag{3.2}$$

where the function $F_\ell(s;\beta,p_a)$ is expressed in terms of an integral over the x^a as

$$F_\ell(s;\beta,p_a) = e^{-\frac{1}{4s}M^2}\int_0^1[dx]e^{-\frac{1}{4s}\left[\sum_a x_a Q_a^2-\left(\sum_a x_a Q_a\right)^2\right]}\times \\ \times\; e^{i\beta\ell\sum_a x_a (Q_a^0)^2}e^{i\sum_a x_a\mathbf{Q}_a\cdot(\theta\mathbf{p}_{\rm np})}\,. \tag{3.3}$$

As in the simpler case of the tadpole of the ϕ^4 theory, whenever $d<2N+2$ we can replace the factor $s^{\frac{d-2N-2}{2}}$ by an integral over $2+2N-d$ extra variables, so we can finally write the diagram in the form of a tree-level exchange of effective fields propagating in $d_\chi^\perp=2N+2-d$ additional "bulk" dimensions:

$$\mathcal{A}(p_1,\ldots,p_N) = \sum_{\ell\in\mathbb{Z}}\int d\mathbf{z}_\perp\frac{f(\ell,p_a,\mathbf{z}_\perp)}{\ell^2\beta^2+(\theta\mathbf{p}_{\rm np})^2+\mathbf{z}_\perp^2}\,, \tag{3.4}$$

where the function $f(p_a,\ell)$ is given by

$$f(\ell,p_a,\mathbf{z}_\perp) = g^{2N}\frac{(N-1)!}{2^{2N}\pi^{d/2}}\mathcal{W}_{\rm NC}\int_0^\infty ds\,F_\ell\left[\frac{s}{\ell^2\beta^2+(\theta\mathbf{p}_{\rm np})^2+\mathbf{z}_\perp^2};\beta,p_a\right].$$

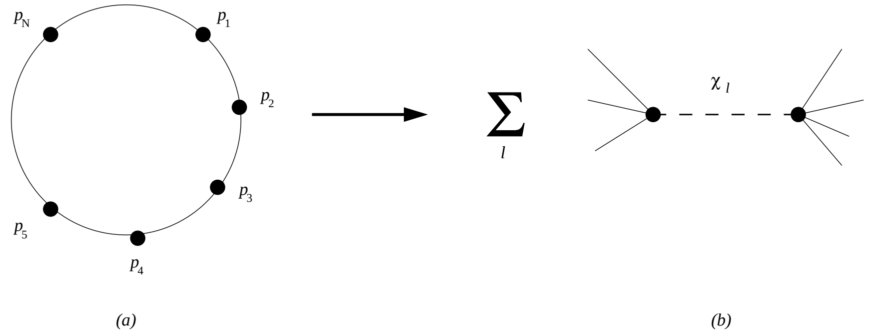

Figure 3: Channel duality for a non-planar thermal loop in ϕ^n non-commutative field theory.

In the same spirit as for the ϕ^4 tadpole one would like to interpret the amplitude (3.4) as a "dual channel" representation of the original loop diagram in terms of a tree-level exchange of χ_ℓ-particles with propagators (2.7), so the function $f(\ell, p_a, \mathbf{z}_\perp)$ would be interpreted as the product of the couplings in figure 3b

$$g_{(\phi^{nN_+})\chi}\, g_{(\phi^{nN_-})\chi} \sim f(\ell, p_a, \mathbf{z}_\perp)\,. \tag{3.5}$$

However, such an identification is rather problematic. Unlike the case of the ϕ^4 tadpole, there seems to be no unambiguous way to define the individual couplings $g_{(\phi^{nN_+})\chi}$ and $g_{(\phi^{nN_-})\chi}$, since their product (3.5) is expressed in terms of a function that does not factorize into the contributions of the two vertices. Moreover, because of the integration over the Feynman parameters in eq. (3.3), the interaction on the two vertices cannot be disentangled, even for massless fields. It is only in the tadpole case ($N = 1$) that the integration over Feynman parameters disappears; the whole loop can then be understood as resulting from the mixing of the field ϕ with an effective χ-field, thus generalizing the result of the previous section to ϕ^n.

Therefore, even if we can formally replace the generic thermal loop by the exchange of an effective χ_ℓ-particle, we cannot assign ordinary Feynman rules to this field, since the total amplitude is expressed as a convolution of the two interaction vertices, and not just as a product, as it is the case of ordinary (and non-commutative) quantum field theory. We summarize this state of affairs by saying that the vertices of the χ_ℓ-fields have relative moduli that must be integrated over.

In principle, the χ_ℓ-fields introduced here could become *bona fide* fields, with standard Feynman rules, when considering only the behaviour of the diagram at singularities of the integral over Feynman parameters. We suspect that this is the precise link between the χ_ℓ-fields defined here and the ψ-fields of refs. [5, 6].

The appearance of moduli in the "dual-channel vertices" will find a string-theory explanation in the next section. First, we shall discuss some special instances in which the formalism simplifies.

3.2 Some special cases at two loops

The previous example seems to indicate that, although in general we can replace non-planar loops in NCFT by tree-level exchanges of an infinite tower of some effective χ_ℓ-fields, in a generic situation the non-local character of this field makes the effective description not very transparent. Here we will further comment on two examples where this effective description is useful.

Let us first consider non-commutative super Yang-Mills (NCSYM) theories at finite temperature. The two-loop free energy density can be written for U(N) NCSYM$_d$ as [16]

$$\begin{aligned}\mathcal{F}(\beta,\theta) = \mathcal{F}(\beta,\theta=0) + \mathcal{C}_{\rm sc}\, g^2 N \Bigg\{ \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \left[\frac{n_b(\mathbf{p})}{\omega_p} + \frac{n_f(\mathbf{p})}{\omega_p}\right]^2 - \\ - \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \int \frac{d\mathbf{q}}{(2\pi)^{d-1}} \left[\frac{n_b(\mathbf{p})}{\omega_p} + \frac{n_f(\mathbf{p})}{\omega_p}\right] \times \\ \times \left[\frac{n_b(\mathbf{q})}{\omega_q} + \frac{n_f(\mathbf{q})}{\omega_q}\right] e^{i\theta(\mathbf{p},\mathbf{q})} \Bigg\}\,,\end{aligned} \tag{3.6}$$

where $\mathcal{C}_{\rm sc} = 16, 4, 1$ for theories with 16, 8 and 4 supercharges, respectively [20], $\omega_p = |\mathbf{p}|$ and $n_{b(f)}(\mathbf{p}) = (e^{\beta|\mathbf{p}|} \mp 1)^{-1}$ are the Bose-Einstein and Fermi-Dirac distribution functions. It is interesting to notice how, for NCSYM theories, the "non-planar" part of the two-loop free energy [the last term in (3.6)] factorizes into the product of two independent loop contributions only linked through the non-commutative phase, in much the same fashion as ϕ^4 NCFT. Following ref. [17] we can now integrate one of these loops to try to spot winding states (figure 4). When $d < 4$ again we can introduce $4-d$ extra variables $\mathbf{z}_\perp$ to write

$$\begin{aligned}\int \frac{d\mathbf{q}}{(2\pi)^{d-1}} \left[\frac{n_b(\mathbf{q})}{\omega_q} + \frac{n_f(\mathbf{q})}{\omega_q}\right] e^{i\theta(\mathbf{p},\mathbf{q})} &= \frac{1}{(2\pi)^2} \sum_{\ell\in\mathbb{Z}} \int d\mathbf{z}_\perp \frac{1+(-1)^{\ell+1}}{\ell^2\beta^2 + (\theta\mathbf{p})^2 + \mathbf{z}_\perp^2} \\ &= \frac{1}{2\pi^2} \sum_{\ell\in\mathbb{Z}} \int d\mathbf{z}_\perp \frac{1}{(2\ell+1)^2\beta^2 + (\theta\mathbf{p})^2 + \mathbf{z}_\perp^2}\,.\end{aligned} \tag{3.7}$$

That is, we find a standard tower of χ_ℓ particles, restricted to *odd* winding numbers. From this expression we learn that, in general, the fermion loops will give rise to χ_ℓ particles with negative norm for even ℓ. For the supersymmetric case, there is a cancellation with the tower coming from the bosonic loop and we find the projection onto odd winding numbers. We shall give a string theory explanation of this phenomenon in the next section.

Notice that the factorization of the two-loop free energy into a contribution of two independent loops is not a general property of any quantum field theory. In particular, for ϕ^3 NCFT the integrand of $\mathcal{F}(\beta,\theta)$ does not have this property even in the massless case. What is special about NCSYM is the fact that many of the

JHEP06(2000)038

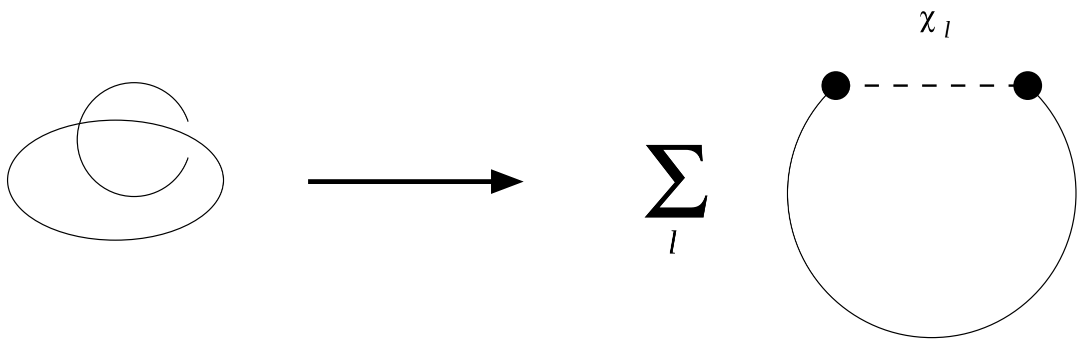

Figure 4: Dual-channel description of the non-planar thermal loop in ϕ^4-like two-loop non-planar vacuum diagram.

individual diagrams contributing to (3.6) have loops with two external insertions, so that at least one Feynman parameter would be needed in order to formally "integrate out" the loop, along the lines of the general discussion above. Yet, the complete two-loop diagram shows factorized form and one can introduce effective χ_ℓ particles with standard Feynman rules (apart from the negative-norm feature in the fermionic case).

The reason behind this simplification is two fold. First, gauge symmetry relates the ϕ^3-like diagrams to the ϕ^4-like diagrams. Second, the theory is massless, so that the effective coupling (2.6) is a momentum-independent pure number, and thus both ϕ^4-like loops are completely disentangled from the kinematical point of view. Therefore, this factorization is, in principle, specific of two-loop diagrams in massless theories whose symmetries can relate all diagrams that contribute to a given physical quantity to ϕ^4-like ones. Another example, considered in [17], is the massless Wess-Zumino model, where supersymmetry plays the relevant role. In the massless limit the Wess-Zumino model reduces itself to a supersymmetric version of ϕ^4 NCFT. Thus the factorization of the two-loop free energy follows from the factorization of the corresponding diagram in ϕ^4 theory under the substitution $n_b(\mathbf{p}) \to n_b(\mathbf{p}) + n_f(\mathbf{p})$. In the NCSYM case it is not supersymmetry, but rather gauge symmetry that plays the simplifying role, because the two-loop factorization is true already for non-supersymmetric non-commutative Yang-Mills theories [16].

4. Windings and closed strings

4.1 Heuristic considerations

Given that many NCFTs derive from open-string theory in background B-fields in the SW limit, it is natural to associate the winding modes of the previous representations to closed strings in intermediate states. Namely, the structure of (2.5) is reminiscent of a closed-string tree-level propagator between the boundary states of a D_{d-1} brane

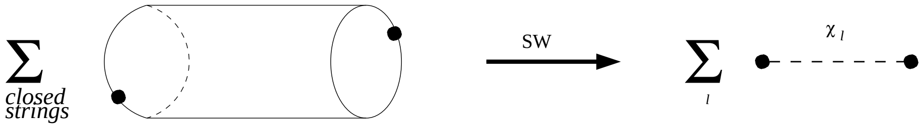

Figure 5: Seiberg-Witten limit of the non-planar two-point function in the closed-string channel.

(figure 5). Heuristically, we expect

$$\begin{aligned}\Pi(\beta,p)_{\rm NP} &= -\sum_{\ell}\int d\mathbf{z}_\perp \frac{|g_\chi(\ell,p,\mathbf{z}_\perp)|^2}{\beta^2\ell^2+(\theta p)^2+\mathbf{z}_\perp^2} \\ &\sim \lim_{\rm SW}\left\langle {\rm D}_{d-1};V_p\left|\frac{1}{\Delta_{cl}}\right|{\rm D}_{d-1};V_p\right\rangle, \end{aligned} \tag{4.1}$$

whereas the planar diagram would be a low-energy limit of $\langle {\rm D}_{d-1};V_p,V_{-p}|\Delta_{cl}^{-1}|{\rm D}_{d-1}\rangle$, for suitably defined boundary states.

There are various pieces of the previous tentative equation that fit nicely. First , the closed-string inverse propagator is

$$\Delta_{cl} = \frac{\alpha'}{2}\left(g^{\mu\nu}p_\mu p_\nu + \frac{\beta^2\ell^2}{4\pi^2\alpha'^2}+M_{cl}^2\right), \tag{4.2}$$

where $g_{\mu\nu}$ is the closed-string or sigma-model metric, to be distinguished from $G_{\mu\nu}$ or open-string metric. The precise relation is defined in [3]:

$$G^{\mu\nu} = \left(\frac{1}{g+2\pi\alpha' B}\right)_S^{\mu\nu}, \qquad \theta^{\mu\nu} = 2\pi\alpha'\left(\frac{1}{g+2\pi\alpha' B}\right)_A^{\mu\nu}, \tag{4.3}$$

where we indicate by the subscripts S and A the symmetric and antisymmetric part, respectively. We can take $g_{\mu\nu}=\delta_{\mu\nu}$ in commutative directions, including the $d_\perp = D-d$ Dirichlet–Dirichlet directions transverse to the ${\rm D}_{d-1}$ brane. On the other hand, in the non-commutative directions, the SW scaling assigns

$$g^{\mu\nu} \longrightarrow -\frac{1}{4\pi^2\alpha'^2}(\theta^2)^{\mu\nu} \tag{4.4}$$

as $\alpha'\to 0$, with $G_{\mu\nu}=\delta_{\mu\nu}$ and $\theta^{\mu\nu}$ fixed. Therefore, the inverse propagator scales like

$$\Delta_{cl} = \frac{1}{8\pi^2\alpha'}\left[\beta^2\ell^2+(\theta p)^2+(2\pi\alpha')^2\left(\mathbf{p}_\perp^2+M_{cl}^2\right)\right] \tag{4.5}$$

and we see that our familiar combination $\beta^2\ell^2+(\theta p)^2$ scales together and *dominates* over the other terms in the SW limit, since $\alpha'^2 M_{cl}^2 \sim \alpha' N_{\rm osc}\to 0$. This is the main evidence for the stringy origin of the winding modes. Indeed, we have Neumann boundary conditions in the thermal circle, and therefore the closed-string cylinder can wind in this direction. On the other hand, there can be no momentum flow

JHEP06(2000)038

through the Neumann directions, unless it is explicitly inserted via the open-string vertices into the boundary states, but there is an arbitrary flow of momentum in Dirichlet-Dirichlet directions. In the non-commutative directions, having a non-zero B-field, there could be a momentum flow induced just by the B-field. The boundary conditions set to zero only a linear combination of momentum and winding numbers. However, since we are assuming a non-compact D-brane in spatial directions, there are no winding modes in these and thus no extra momentum flow induced by the B-field.

If we are willing to naively neglect the nominally subleading terms in (4.5) we can almost get (4.1) with the coupling of the χ_ℓ-fields defined through the "sum rule" over all closed-string fields $|\Psi\rangle$ (the oscillator excitations)

$$|g_{\chi\phi}|^2 \longrightarrow \sum_{\Psi} \langle \mathrm{D}_{d-1}; V_p|\Psi\rangle \, \langle\Psi|\, \mathrm{D}_{d-1}; V_p\rangle \tag{4.6}$$

in the SW limit, perhaps with appropriate powers of α' in front. According to this picture, the low-energy χ_ℓ-fields *are not* some low-lying closed-string modes that fail to decouple. In fact, the whole infinite tower of closed-string modes fails to decouple, but the interaction with the boundary states defines an effective coupling for the χ_ℓ-field, which represents the coherent exchange of an infinite number of closed-string excitations. Formally, the SW limit squeezes the complete tower of string excited states into an approximately continuous band with respect to the gap of the winding modes

$$\frac{\text{Oscillator gap}}{\text{Winding gap}} \sim \frac{\alpha'}{\beta^2} \longrightarrow 0\,. \tag{4.7}$$

This is an interesting compromise between the general lore that the closed-string channel should be intractable whenever the open-string channel is simple [11], and the factual existence of the dual-channel representation in terms of the χ_ℓ-fields.

Actually, the sum rule (4.6) is too naive. The first indication that something is missing in (4.6) is the fact that a naive attempt to associate the $\mathbf{z}_\perp$ degrees of freedom with Dirichlet-Dirichlet momenta $\mathbf{p}_\perp$ in the D-brane codimension fails quantitatively, because in general $d_\perp \neq d_\chi^\perp = 2 + 2N - d$ for a D_{d-1} brane. The resolution of this puzzle amounts to recognizing that one cannot simply neglect $\alpha'^2 M_{cl}^2$ in the closed-string propagator, as compared with $\beta^2\ell^2$, even in the low-energy SW limit, because there are an infinite number of states contributing to the sum. In other words, the truncated sum rule (4.6) is not convergent in general.

The second reason for concern lies in the definition of the low-energy effective coupling $g_{\chi\phi}$ as a proper effective vertex. In the full string-theory diagram, the boundary states with open-string insertions have moduli (the Koba-Nielsen parameters) that must be integrated over. Therefore, the stringy diagram does not have in general the structure of an ordinary tree-level exchange, since both vertices are convoluted in an integral over Koba-Nielsen parameters. Only at the boundaries of the moduli space, when the string diagram degenerates into proper field-theory diagrams, does one find standard Feynman rules. This means that the correct sum rule replacing (4.6) must hold for the integrand over moduli space (including the Koba-Nielsen moduli).

4.2 Open-string channel

In the remainder of this section we obtain the correct sum rule by careful consideration of the general one-loop open-string diagram. This is a weighted sum over spin structures $\{\sigma\}$, each one given by a path integral on the annulus, with arbitrary vertex operator insertions on both the inner $(-)$ and outer $(+)$ boundaries:

$$\begin{aligned}\mathcal{A} &= \sum_\sigma C_\sigma \mathcal{A}_\sigma = \sum_\sigma C_\sigma \int_0^\infty \frac{d\tau}{2\tau} \int_0^\tau [dy^\pm] \left\langle \prod_{y^\pm} V_\phi(p^\pm, y^\pm) \right\rangle_\sigma \\ &= (\alpha')^{\mathcal{N}_0}\, G_s^{N/2} \sum_\sigma C_\sigma \int_0^\infty \frac{d\tau}{2\tau} Z(\tau)_\sigma \int_0^\tau [dy^\pm]\, \mathcal{V}(p^\pm, y^\pm, \tau)_\sigma\,, \end{aligned} \tag{4.8}$$

where τ is the modulus of the annulus, $y^\pm$ are the Koba-Nielsen parameters in each boundary, $\mathcal{V}(p^\pm, y^\pm, \tau)$ is the normalized correlator of vertex operators in the spin structure σ, and $Z(\tau)_\sigma$ is the normalization, i.e. the path integral without vertex insertions. The power of G_s comes from the normalization of the vertex operators with the open-string coupling defined in [3]

$$G_s = g_s \left[\frac{\det G}{\det\,(g + 2\pi\alpha' B)}\right]^{1/2} = \frac{(\alpha')^{\frac{4-d}{2}}}{(2\pi)^{d-3}}\, g_{\rm YM}^2\,. \tag{4.9}$$

The complete amplitude has an appropriate power of the Regge slope $\mathcal{N}_0$, such that the SW limit with fixed $g_{\rm YM}$ leads to the field-theoretical expression of the amplitude. This depends on the overall dimension of the amplitude and the number of insertions.

The function $\mathcal{V}(p^\pm, y^\pm, \tau)$ is a polynomial in external field polarizations, the fermionic Green function $\mathcal{G}_{\rm F}$ and derivatives of the bosonic Green function $\mathcal{G}_{\rm B}$ on the annulus with appropriate B-dependent boundary conditions, times the contraction of the tachyonic part of the vertices

$$\mathcal{V}(p^\pm, y^\pm, \tau) = \mathcal{P}(\phi, p^\pm, \mathcal{G}_{\rm F}, \partial\mathcal{G}_{\rm B})\, e^{p\cdot\mathcal{G}_{\rm B}\cdot p}\,. \tag{4.10}$$

This Green function can be parametrized completely by the open-string metric $G^{\mu\nu}$ and noncommutativity parameters $\theta^{\mu\nu}$, except for a single constant term from the purely bosonic component [21], which contributes to non-planar diagrams and depends explicitly on the sigma model metric $g_{\mu\nu}$. If we separate this contribution from (4.10) we can write

$$\mathcal{V}(p^\pm, y^\pm, \tau) = \overline{\mathcal{V}}(p^\pm, y^\pm, \tau)\ \exp\left[-\frac{\alpha'\pi^2}{\tau} g^{\mu\nu}(p_\mu p_\nu)_{\rm cyl}\right], \tag{4.11}$$

where $p_{\rm cyl} = \sum p^+ = -\sum p^-$ is the total momentum circulating in the closed-string channel. We recognize this term as the standard kinetic term in Δ_{cl} (4.2).

If we assume, for simplicity, that the external insertions are space-time bosons with no thermal frequency, i.e. we have a purely static bosonic correlator, then the world-sheet partition sum can be written directly in operator form as

$$Z(\tau)_\sigma = \frac{1}{\mathrm{Vol}_G} \; \mathrm{Tr}_{\mathrm{open}} \;\; \mathcal{S}_\sigma \; e^{-\tau\,\Delta_\sigma} \,, \tag{4.12}$$

where $\mathcal{S}_\sigma$ is a piece of the GSO projector, $\pm\frac{1}{2}$ or $\pm\frac{1}{2}(-1)^F$ depending on the spin structure, Δ_σ is the open-string world-sheet hamiltonian, and we have normalized by the volume in the open-string metric $G_{\mu\nu}$. The temperature-dependent part of (4.12) is unaffected by the B-field, as long as we keep $B_{0i} = 0$. Therefore, it has the form

$$\Delta(\beta)_\sigma = \alpha' \left(p^0\right)^2 = \alpha' \frac{4\pi^2 n_\sigma^2}{\beta^2} \,, \tag{4.13}$$

with n_σ integer in those spin structures running space-time bosons in the loop, and half-integer in those running space-time fermions. From here we can read off the relation between the annulus modular parameter and the low-energy Schwinger parameter of the field theory expressions in eq. (3.1) of the previous section; it is

$$t = \alpha' \, \tau \,. \tag{4.14}$$

For the comparison with the low-energy expression, it is useful to work with Koba-Nielsen parameters normalized to unity, $y = \tau\, x$, so that a further factor of $\tau^N = (t/\alpha')^N$ appears for a total of N insertions. Consistency of the SW low-energy limit requires that, for an appropriate choice of $\mathcal{N}_0$, the field-theoretical amplitude is obtained as

$$\begin{aligned}\mathcal{A}_{\mathrm{NCFT}} = \lim_{\mathrm{SW}} \, (\alpha')^{\mathcal{N}_0 + \frac{4-d}{4} N} \, g_{\mathrm{YM}}^N \, \times \\ \times \int_0^\infty dt \, t^{N-1} \sum_\sigma C_\sigma \int_0^1 [dx^\pm] Z\left(\frac{t}{\alpha'}\right)_\sigma \, \mathcal{V}\left(p^\pm, x^\pm, \frac{t}{\alpha'}\right)_\sigma \,.\end{aligned} \tag{4.15}$$

The precise details of this limit in various examples of the bosonic theory at zero temperature can be found in recent papers [8]–[11], [13]. The important feature is that the SW limit is dominated by massless open strings (in the bosonic examples one is forced to discard the open-string tachyon by hand). Thus, in comparing with (3.1), we must set $M = 0$ and interpret the normalized Koba-Nielsen parameters $x^\pm = y^\pm/\tau$ as Feynman parameters of the field-theory diagram.

On general grounds, the massless open-string dominance means that we do not expect the closed-string channel expression to be simple, in the sense of being saturated by a finite number of closed-string fields.

4.3 The closed-string channel sum rule

Ideally, we would like to specify explicitly, in the closed-string Fock space, the boundary states that appear in eq. (4.1). This is a very complicated task in general, and can be carried out in detail only for the "vacuum" boundary states $|\mathrm{D}_{d-1}\rangle$ without open-string vertex insertions. On the other hand, we can obtain explicitly the overlap in (4.1), by direct modular transformation of the open-string channel expression (4.8).

In order to keep track of the right normalization of winding modes, we perform a Poisson resummation of the discrete frequency sums:

$$\frac{1}{\beta}\sum_{n_\sigma} e^{-\tau\Delta(\beta)} = (4\pi\alpha'\tau)^{-1/2}\sum_{\ell\in\mathbb{Z}} e^{-\frac{\beta^2\ell^2}{4\alpha'\tau}}\,(-1)^{\ell\mathbf{F}_\sigma}\,, \tag{4.16}$$

where $\mathbf{F}_\sigma$ is the *space-time* fermion number in the open-string channel. It is correlated with the closed-string sector in the cylinder channel in such a way that $\mathbf{F}=0$ corresponds to the NS-NS sector of the Dp-brane boundary state, whereas $\mathbf{F}=1$ leads to the R-R exchange [22]. Therefore, we have found that the winding modes of closed strings in the R-R sector are weighted by the so-called Atick-Witten phase $-(-1)^\ell$ [23], the extra minus sign coming from the GSO projection, or more elementarily, from the overall minus sign of fermion loops in space-time. We recognize in the phase $(-1)^{\ell+1}$ the effect pointed out in eq. (3.7). Namely, fermionic loops in the "open-string channel" lead to phases in winding modes in the "closed-string channel". In particular, in the supersymmetric case we also find a projection onto odd winding numbers in the full string theory expression. The closed-string interpretation also explains the "negative norm" of the tower of even $\chi_{2\ell}$-fields coming from a fermion loop; it is just an effect of the D-brane carrying "axionic" charge with respect to these fields.

In view of the closed-string propagator in (4.2), the appropriate modular transformation to obtain $\exp(-\tau_2\,\Delta_{cl})$ is $\tau_2=2\pi^2/\tau$. The non-trivial piece of the overlap (4.1) is that of the oscillator degrees of freedom. We define

$$\begin{aligned} Z\left(\frac{2\pi^2}{\tau_2}\right)^{\rm osc}_\sigma \overline{\mathcal{V}}\left(x^\pm,p^\pm,\frac{2\pi^2}{\tau_2}\right)_\sigma &= \left(\frac{\tau_2}{\pi}\right)^{\frac{2-D}{2}+N}\times \\ &\times\left\langle \mathrm{D}_{d-1};V_{p^+},x^+\middle|e^{-\tau_2\Delta_{cl}^{\rm osc}}\middle|\mathrm{D}_{d-1};V_{p^-},x^-\right\rangle_\sigma\,, \end{aligned} \tag{4.17}$$

where

$$\Delta_{cl}^{\rm osc} = \frac{\alpha'}{2}M_{cl}^2\,.$$

Numerical constants have been absorbed in the definition of the boundary states. The modular anomaly, depending on the total dimension where the string oscillates ($D=10$ for superstrings), is captured by looking at the case without insertions, where there exists an explicit construction of the boundary states. Taking into account the factor of $\tau_2^{\frac{d-1}{2}}$ from the integral over world-volume momenta in the evaluation of $Z(\tau)_\sigma$, we find the total moduli-space measure to be

$$d\tau_2 \, \tau_2^{-d_\perp/2} \, [dx^\pm] \, . \tag{4.18}$$

The factor of $\tau_2^{-d_\perp/2}$ can be "integrated in" by introducing explicitly the integral over the $(D-d)$-dimensional transverse momenta $\mathbf{p}_\perp$, which replaces $M_{cl}^2 \to M_{cl}^2 + \mathbf{p}_\perp^2$ and then we have a standard measure for a propagator, as in (4.1). Notice that the power of τ_2^N in (4.17) is crucial in obtaining (4.18), so that the only τ_2-dependence of the overlap is in the world-sheet evolution operator. We see explicitly how the proper counting of transverse dimensions, as read-off from the powers of Schwinger parameters, is working fine, thanks to the modular anomaly in (4.17).

Collecting all terms, we can write a closed-string channel expression for the total amplitude in the SW limit. In order to make contact with the expressions in section 3, we define a dual Schwinger parameter with mass squared dimension

$$s = \frac{\tau_2}{8\pi^2\alpha'} \, ,$$

in terms of which we have

$$\begin{aligned} \mathcal{A}_{\rm NCFT} = \lim_{\rm SW} \; & g_{\rm YM}^N \, (\alpha')^{\mathcal{N}_0 + \frac{4-d}{4} N} \times \\ & \times \int_0^\infty ds \; s^{-d_\perp/2} \int_0^1 [dx^\pm] \sum_\sigma C_\sigma \sum_{\ell\in\mathbb{Z}} e^{-s[\beta^2\ell^2 + (\theta p)^2]} \, U_{\ell,\sigma} \times \\ & \times \left\langle {\rm D}_{d-1}; V_{p^+}, x^+ \middle| e^{-s(2\pi\alpha' M_{cl})^2} \middle| {\rm D}_{d-1}; V_{p^-}, x^- \right\rangle_\sigma \end{aligned} \tag{4.19}$$

with $U_{\ell,\sigma}$ the Atick-Witten phase. We can recast this expression in the form of eq. (3.2)

$$\begin{aligned} \mathcal{A}_{\rm NCFT} = \lim_{\rm SW} \; & \mathcal{W}_{\rm NC} \, g_{\rm YM}^N \int_0^\infty ds \, s^{-N + \frac{d-2}{2}} \times \\ & \times \sum_{\ell\in\mathbb{Z}} e^{-s[\beta^2\ell^2 + (\theta p)^2]} \int_0^1 [dx^\pm] \; F_\ell(s, p^\pm, x^\pm) \, , \end{aligned} \tag{4.20}$$

up to numerical constants. This expression implies a "sum rule" for the function $F_\ell(s,p,x)$:

$$\begin{aligned} \mathcal{W}_{\rm NC} F_\ell \left(s, p^\pm, x^\pm\right) = \lim_{\rm SW} \; & (\alpha')^{\mathcal{N}_0 + \frac{4-d}{4} N} \; s^{\frac{2-D}{2} + N} \times \\ & \times \sum_\sigma C_\sigma \, U_{\ell,\sigma} \sum_\Psi \left\langle {\rm D}_{d-1}; V_{p^+}, x^+ \middle| \Psi \right\rangle_\sigma \left\langle \Psi \middle| e^{-s(2\pi\alpha' M_\Psi)^2} \middle| \Psi \right\rangle_\sigma \times \\ & \times \left\langle \Psi \middle| {\rm D}_{d-1}; V_{p^-}, x^- \right\rangle_\sigma \, , \end{aligned} \tag{4.21}$$

where the sum over closed-string states $|\Psi\rangle$ runs over all oscillator degrees of freedom of the closed strings in the bulk. In comparing (4.21) with the field-theoretical expression in (3.3), we must take into account the fact that $M = 0$ and that (3.3) was derived for a purely bosonic loop, hence there is no non-trivial Atick-Witten phase in (3.3). Furthermore, (4.21) was derived under the assumption that external states were bosonic and *static*, i.e. external momenta have vanishing time components. That explains the absence of the phase $\exp[i\beta\ell\sum_a x_a(Q_a^0)^2]$ in (4.21).

We conclude this subsection with some observations on the interpretation of (4.21):

(i) In terms of the dimensionless closed-string modulus, the SW limit in (4.21) takes $\tau_2 \to 0$. In this region of moduli space, the infinite tower of closed-string fields contributes to the sum rule, which is by no means saturated by a few closed-string fields. This was already obvious from the fact that the open-string channel expression *was* saturated by massless open strings.

(ii) The sum rule (4.21) replaces the naive one in (4.6). One of the defects of (4.6), the mismatch between the powers of the Schwinger parameter and the true number of transverse dimensions of the brane, is resolved by noticing that the sum rule includes a non-trivial power of $s^{\frac{D-2}{2}}$, together with an explicit exponential kernel, which gives back the field-theoretical measure of (3.2) in the SW limit, and ensures the convergence of the sum over closed-string states. In fact, since this limit takes $\tau_2 \sim \alpha' s \to 0$, the way to evaluate the infinite sum over states in (4.21) is to perform a modular transformation back to the open-string variables. In this process we get the appropriate powers of the Schwinger parameter from the modular anomaly of the oscillator traces (Jacobi's theta functions).

This discussion makes also manifest the formal character of the extra "bulk dimensions" $d_\chi^\perp = 2N + 2 - d$ of the winding fields χ_ℓ, something already made clear by the fact that $d_\chi^\perp$ depends on the number of insertions in the loop. We see that, in those models with a string-theory embedding, there is a "bulk" codimension $d_\perp = D - d$, but its relation with $d_\chi^\perp$ is rather indirect.

(iii) Another deficiency of (4.6), the absence of Koba-Nielsen parameters, is remedied in (4.21). The interpretation of the full diagram in the NCFT, as a tree-level exchange of χ_ℓ-fields, was correct provided we make a further convolution of the vertices with Feynman parameters. We now understand this feature as a residue of the full string picture, since Koba-Nielsen parameters map consistently to Feynman parameters in the SW limit. Therefore, the χ_ℓ-field picture of the NCFT mimics closely the structure of the closed-string channel in the full string theory.

(iv) The sum rule (4.21) holds for the integrand of the moduli-space integral. Therefore, it holds independently of the possible occurrence of open- or closed-

JHEP06(2000)038

channel tachyons in the full string theory. This is in contrast with (4.6), which would be invalidated by open-string tachyons, and perhaps also by closed-string tachyons. It would be very interesting to study in detail particular examples, so as to see the interplay between the various open/closed tachyons that could appear, including the finite-temperature Hagedorn tachyon.

(v) Our discussion is tailored to the case of thermal amplitudes. However, it is clear that the broad features generalize to other toroidal compactifications with various degrees of supersymmetry, and also to orbifold models, such as those considered in ref. [12]. In particular, it would be interesting to find the general BPS conditions for the sum rule to be saturated by a *finite* number of closed-string fields.

4.4 An illustrative example

Unlike eq. (4.6), the sum rule (4.21) is valid point by point in the $(s, x^{\pm})$ moduli space. As a consequence, it is well defined even for tachyonic theories for which the integrated expressions would diverge because of the contribution coming from the moduli space boundaries. This being so, we can illustrate our sum rule (4.21) by considering the simplest possible example and take the two-point function of open-string tachyons on a D_{d-1} brane of the $D = 26$ critical bosonic string theory. In order to avoid unnecessary complications, we will consider the static amplitude where incoming states do not carry time-components of the momenta. Thus, the amplitude in the open-string channel can be written as [9, 10, 13]

$$\begin{aligned} \mathcal{A}(p,-p)_{\text{tachyon}} = G_s \int_0^\infty \frac{d\tau}{2\tau} (4\pi\alpha'\tau)^{-d/2} \left[\eta\left(\frac{i\tau}{2\pi}\right)\right]^{-24} \times \\ \times \sum_{\ell\in\mathbb{Z}} e^{-\frac{\ell^2\beta^2}{4\alpha'\tau}}\; e^{-\frac{\alpha'\pi^2}{\tau} p_\mu (g^{\mu\nu} - G^{\mu\nu}) p_\nu}\, \tau^2 \times \\ \times \int_0^1 dx^{\pm} \left| 2\pi e^{-\frac{x_{12}^2}{2\tau}} \frac{\theta_2\left(\frac{ix_{12}\tau}{2\pi}\middle|\frac{i\tau}{2\pi}\right)}{\theta_1'\left(0\middle|\frac{i\tau}{2\pi}\right)} \right|^{-2}, \end{aligned}$$

where we have defined $x_{12} \equiv x^+ - x^-$. Comparing this expression with eqs. (4.8) and (4.10), we can read both $Z(\tau)^{\mathrm{osc}}$ and $\overline{\mathcal{V}}(p, x^{\pm}, \tau)$, in terms of which the overlap (4.17) is expressed:

$$\begin{aligned} Z(\tau)^{\mathrm{osc}} &= \left[\eta\left(\frac{i\tau}{2\pi}\right)\right]^{-24}, \\ \overline{\mathcal{V}}(p^{\pm}, x^{\pm}, \tau) &= e^{\frac{\alpha'\pi^2}{\tau} p_\mu G^{\mu\nu} p_\nu} \left| 2\pi e^{-\frac{x_{12}^2}{2\tau}} \frac{\theta_2\left(\frac{ix_{12}\tau}{2\pi}\middle|\frac{i\tau}{2\pi}\right)}{\theta_1'\left(0\middle|\frac{i\tau}{2\pi}\right)} \right|^{-2}. \end{aligned}$$

Switching from τ to the closed-string modular parameter $\tau_2 = 2\pi^2/\tau$, and performing the inversion on the modular functions, we find for the partition function of the oscillators

$$Z\left(\frac{2\pi^2}{\tau_2}\right)^{\rm osc} = \left(\frac{\tau_2}{\pi}\right)^{-12}\left[\eta\left(\frac{i\tau_2}{\pi}\right)\right]^{-24},$$

whereas the function $\overline{\mathcal{V}}(p^\pm, x^\pm, \tau)$ is written

$$\overline{\mathcal{V}}\left(p, x^\pm, \frac{2\pi^2}{\tau_2}\right) = \left(\frac{\tau_2}{\pi}\right)^2 e^{\frac{1}{2}\alpha'\tau_2 p_\mu G^{\mu\nu}p_\nu}\left|2\pi\frac{\theta_4\left(x_{12}|\frac{i\tau_2}{\pi}\right)}{\theta_1'\left(0\,|\frac{i\tau_2}{\pi}\right)}\right|^{-2}.$$

Using eq. (4.17) we can now obtain the expression for the overlap, namely

$$\left\langle {\rm D}_{d-1}; V_{p^+}, x^+\left|e^{-\tau_2\Delta_{cl}^{\rm osc}}\right|{\rm D}_{d-1}; V_{p^-}, x^-\right\rangle = e^{\frac{1}{2}\tau_2}\left[\eta\left(\frac{i\tau_2}{\pi}\right)\right]^{-18}\left[\theta_4\left(x_{12}\left|\frac{i\tau_2}{\pi}\right.\right)\right]^{-2},$$

where we have used the on-shell condition for the external tachyons, $p_\mu G^{\mu\nu}p_\nu = 1/\alpha'$ and also the relation $\theta_1'(0|\tau) = 2\pi\eta^3(\tau)$. Actually, the modular functions can be rewritten using their product representations. Expanding the resulting infinite products in power series of $e^{-2\tau_2}$, we finally arrive at

$$\begin{aligned}
&\left\langle {\rm D}_{d-1}; V_{p^+}, x^+\left|e^{-\tau_2\Delta_{cl}^{\rm osc}}\right|{\rm D}_{d-1}; V_{p^-}, x^-\right\rangle = \\
&\quad = e^{2\tau_2}\prod_{k=1}^{\infty}\left(1-e^{-2k\tau_2}\right)^{-20}\left|1-e^{-(2k-1)\tau_2}e^{2\pi i x^+}e^{-2\pi i x^-}\right|^{-4} \\
&\quad = \sum_{n=0}^{\infty}\rho_{\rm D}(n)\, C_n^+(x^+)\, e^{-\frac{1}{2}\alpha'\tau_2 M_n^2}\, C_n^-(x^-)^*\,,
\end{aligned} \tag{4.22}$$

where $M_n^2 = \frac{4}{\alpha'}(n-1)$ is the mass of the level-n oscillator states, and $\rho_{\rm D}(n)$ is the level-density of those states with non-vanishing coupling to the ${\rm D}_{d-1}$ brane. Notice that, because of the structure of the product representation for the modular functions, the coefficient of $e^{-2(n-1)\tau_2}$ in the series always factorizes into contributions from the two different boundaries, $C_n^\pm(x^\pm)$, weighted by the level-density, $\rho_{\rm D}(n)$. Comparing this expression with the sum rule (4.21) we read-off the couplings of a level-n closed-string state $|\Psi_n\rangle$ to the boundary state with an external open-string tachyon insertion

$$\left\langle {\rm D}_{d-1}; V_{p^+}, x^\pm\middle|\Psi_n\right\rangle = C_n^\pm(x^\pm)\,. \tag{4.23}$$

In this example we explicitly see how the coupling between $|\Psi_n\rangle$ and the boundary state with a tachyon insertion is in general non vanishing for all values of n and thus *all* closed-string oscillator levels run in the cylinder. As a consequence, the effective χ_ℓ-fields cannot be seen as some undecoupled winding string state in the SW limit, but rather as a superposition of all massive closed-string modes with coherent couplings to the elementary quanta of the NCFT.

5. Concluding remarks

In the present paper we have tried to identify the stringy connection of the recently conjectured winding modes emerging in NCFTs [17]. We have seen how a thermal loop in NCFT can be represented in a "dual-channel" picture as an infinite tower of tree-level exchanges of some effective fields χ_ℓ with masses proportional to $|\beta\ell|$ ($\ell \in \mathbb{Z}$) and kinetic term $\partial \circ \partial$ in the effective action. The scaling of the masses of these fields with the length of the euclidean time suggests a winding-mode interpretation for them.

In many respects these fields are similar to the ψ-fields introduced in refs. [5, 6]. It is important to notice, however, that there are a number of differences. First of all, in the thermal case we have not just one, but an infinite tower of effective fields replacing the thermal loop in non-planar amplitudes. As a consequence, one is able to replace the *whole* thermal loop by a tree-level exchange of these fields and not just the high energy part as in [5]. Most importantly, the χ_ℓ-fields have non-standard Feynman rules. If the nonlocality in NCFT is just encoded in a non-polynomial dependence of the interaction vertices on the incoming momenta, in the perturbation theory for the effective χ_ℓ-fields the interaction vertices are convoluted in finite-dimensional integrals over the Feynman parameters of the original diagram.

Actually, both features, winding-like masses and integration over the relative moduli of the interaction vertices, strongly suggest a stringy interpretation. We have found that such an interpretation exists in those models that can be obtained from a D-brane theory in the presence of a constant B-field in the SW limit.[4] In this case, we found that the scaling of the masses of the χ_ℓ-fields with the length of the thermal circle is a residue of winding closed-string states, whereas the convolution over the vertex moduli descends from the integration over the Koba-Nielsen moduli of the D-brane boundary states.

It is however important to notice that the "winding states" identified in [17] have no simple interpretation in terms of *individual* string states that fail to decouple in the SW limit. On the contrary, the χ_ℓ-fields have to be considered formal devices to represent a coherent coupling of an infinite number of closed-string states. These fields have effective couplings to the ordinary fields that can be derived from the elementary coupling of closed strings to D-brane boundaries via "sum rules" involving the full tower of closed-string oscillator modes in the bulk. This is a rather unusual picture, and it is essentially telling us that the SW limit is not an ordinary low-energy limit in the closed-string channel, since *all* massive states are squeezed below the gap of the winding modes. As a consequence, the resulting NCFTs present a degenerate version of the open/closed string dual-

[4] The phenomenon of UV/IR mixing is also present in non-relativistic NCFTs, for which no obvious embedding into a string theory seems to exist [24].

ity of the original string theory: the ordinary "open" representation of the Feynman diagram in NCFT and the "closed" dual channel in terms of the winding χ_ℓ-fields.

Acknowledgments

It is a pleasure to thank Luis Alvarez-Gaumé, Jan de Boer, Karl Landsteiner, Esperanza López, Niels Obers and Eliezer Rabinovici for useful discussions. G.A. warmly thanks the Spinoza Instituut and specially Gerard 't Hooft for hospitality. The work of G.A. is supported by EEC under TMR contract ERBFMRX-CT96-0045. J.L.F.B is on leave from the University of Santiago de Compostela, Spain. J.G. is partially supported by CERN and by Spanish Science Ministry Grant AEN98-043 and Catalan Science Foundation Grant GC 1998 SGR. The work of M.A.V.-M. has been supported by FOM (*Fundamenteel Onderzoek van de Materie*), by University of the Basque Country Grants UPV 063.310-EB187/98 and UPV 172.310-G02/99 and by Spanish Science Ministry Grant AEN99-0315. M.A.V.-M. also wishes to thank the Spinoza Instituut at the University of Utrecht and CERN Theory Division for hospitality during the completion of this work.

References

[1] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[2] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[4] A. Gonzalez-Arroyo and C.P. Korthals-Altes, *Reduced model for large-N continuum field theories*, *Phys. Lett.* **B 131** (1983) 396;
T. Filk, *Divergencies in a field theory on quantum space*, *Phys. Lett.* **B 376** (1996) 53;
J.C. Várilly and J.M. Gracia-Bondia, *On the ultraviolet behaviour of quantum fields over noncommutative manifolds*, *Int. J. Mod. Phys.* **A 14** (1999) 1305 [hep-th/9804001];
C.P. Martín and D. Sanchez-Ruiz, *The one-loop UV divergent structure of* U(1) *Yang-Mills theory on noncommutative* R^4, *Phys. Rev. Lett.* **83** (1999) 476 [hep-th/9903077];
M.M. Sheikh-Jabbari, *Renormalizability of the supersymmetric Yang-Mills theories on the noncommutative torus*, *J. High Energy Phys.* **06** (1999) 015 [hep-th/9903107];
T. Krajewski and R. Wulkenhaar, *Perturbative quantum gauge fields on the noncommutative torus*, *Int. J. Mod. Phys.* **A 15** (2000) 1011 [hep-th/9903187];

D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, hep-th/9908056;
I.Y. Aref'eva, D.M. Belov and A.S. Koshelev, *Two-loop diagrams in noncommutative Φ_4^4 theory*, *Phys. Lett.* **B 476** (2000) 431 [hep-th/9912075];
M. Hayakawa, *Perturbative analysis on infrared aspects of noncommutative QED on R^4*, *Phys. Lett.* **B 478** (2000) 394 [hep-th/9912094]; *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R^4*, hep-th/9912167;
I.Y. Aref'eva, D.M. Belov and A.S. Koshelev, *A note on UV/IR for noncommutative complex scalar field*, hep-th/0001215;
A. Matusis, L. Susskind and N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, hep-th/0002075.

[5] S. Minwalla, M.V. Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, hep-th/9912072.

[6] M.V. Raamsdonk and N. Seiberg, *Comments on noncommutative perturbative dynamics*, *J. High Energy Phys.* **03** (2000) 035 [hep-th/0002186].

[7] A. Hashimoto and N. Itzhaki, *Non-commutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett.* **B 465** (1999) 142 [hep-th/9907166];
J.M. Maldacena and J.G. Russo, *Large-N limit of non-commutative gauge theories*, *J. High Energy Phys.* **09** (1999) 025 [hep-th/9908134];
M. Alishahiha, Y. Oz and M.M. Sheikh-Jabbari, *Supergravity and large-N noncommutative field theories*, *J. High Energy Phys.* **11** (1999) 007 [hep-th/9909215];
A. Hashimoto and N. Itzhaki, *On the hierarchy between non-commutative and ordinary supersymmetric Yang-Mills*, *J. High Energy Phys.* **12** (1999) 007 [hep-th/9911057];
T. Harmark and N.A. Obers, *Phase structure of non-commutative field theories and spinning brane bound states*, *J. High Energy Phys.* **03** (2000) 024 [hep-th/9911169].

[8] O. Andreev and H. Dorn, *Diagrams of noncommutative Φ^3 theory from string theory*, hep-th/0003113.

[9] Y. Kiem and S. Lee, *UV/IR mixing in noncommutative field theory via open string loops*, hep-th/0003145.

[10] A. Bilal, C.-S. Chu and R. Russo, *String theory and noncommutative field theories at one loop*, hep-th/0003180.

[11] J. Gomis, M. Kleban, T. Mehen, M. Rangamani and S. Shenker, *Noncommutative gauge dynamics from the string worldsheet*, hep-th/0003215.

[12] A. Rajaraman and M. Rozali, *Noncommutative gauge theory, divergences and closed strings*, *J. High Energy Phys.* **04** (2000) 033 [hep-th/0003227].

[13] H. Liu and J. Michelson, *Stretched strings in noncommutative field theory*, hep-th/0004013.

[14] J.L. F. Barbón and E. Rabinovici, *On $1/N$ corrections to the entropy of noncommutative Yang-Mills theories*, *J. High Energy Phys.* **12** (1999) 017 [hep-th/9910019].

[15] R.-G. Cai and N. Ohta, *On the thermodynamics of large-N non-commutative super Yang-Mills theory*, *Phys. Rev.* **D 61** (2000) 124012 [hep-th/9910092].

[16] G. Arcioni and M.A. Vázquez-Mozo, *Thermal effects in perturbative noncommutative gauge theories*, *J. High Energy Phys.* **01** (2000) 028 [hep-th/9912140].

[17] W. Fischler, J. Gomis, E. Gorbatov, A. Kashani-Poor, S. Paban and P. Pouliot, *Evidence for winding states in noncommutative quantum field theory*, *J. High Energy Phys.* **05** (2000) 024 [hep-th/0002067].

[18] W. Fischler, E. Gorbatov, A. Kashani-Poor, R. McNees, S. Paban and P. Pouliot, *The interplay between θ and t*, *J. High Energy Phys.* **06** (2000) 032 [hep-th/0003216].

[19] J.I. Kapusta, *Thermal field theory*, Cambridge University Press, 1989.

[20] M.A. Vázquez-Mozo, *A note on supersymmetric Yang-Mills thermodynamics*, *Phys. Rev.* **D 60** (1999) 106010 [hep-th/9905030].

[21] A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, *Open strings in background gauge fields*, *Nucl. Phys.* **B 280** (1987) 599.

[22] M.A. Vázquez-Mozo, *Open string thermodynamics and D-branes*, *Phys. Lett.* **B 388** (1996) 494 [hep-th/9607052].

[23] J.J. Atick and E. Witten, *The hagedorn transition and the number of degrees of freedom of string theory*, *Nucl. Phys.* **B 310** (1988) 291.

[24] J. Gomis, K. Landsteiner and E. López, *Nonrelativistic noncommutative field theory and UV/IR mixing*, hep-th/0004115.